- Lagrange multipliers: Finding extrema of f on $g^{-1}(c)$.

$$\left[\begin{array}{l} a \text{ is a local extrema of } f \text{ on } g^{-1}(c) \\ \nabla g(a) \neq 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} \nabla f(a) = \lambda \nabla g(a) \\ g(a) = c \end{array}\right.$$

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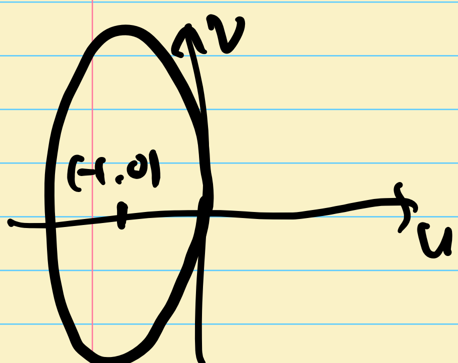
- $g(x,y) = Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F$

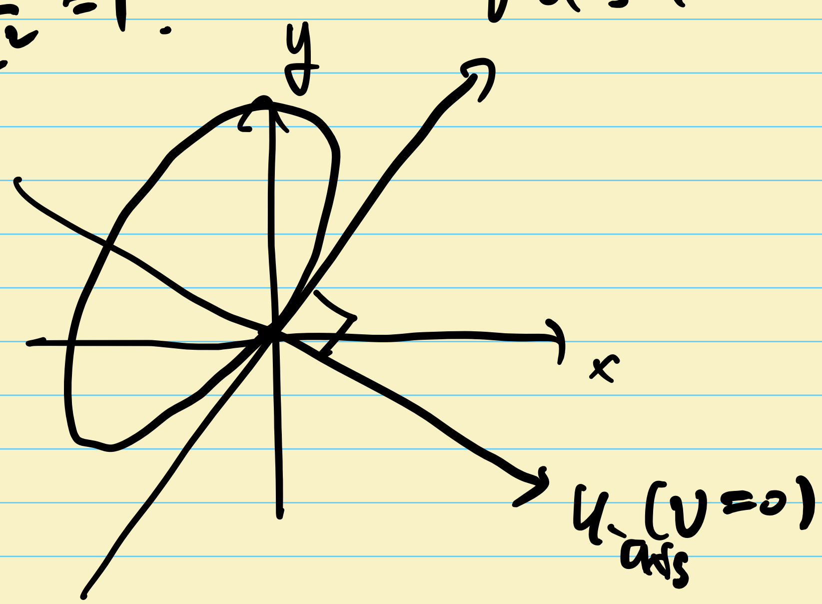
Fact: By a change of coordinates, any quadratic constraint $g(x,y) = c$ can be transformed to one of: hyperbola, parabola, ellipse, degenerate cases.

eg $g(x,y) = 17x^2 - 12xy + 8y^2 + 16\sqrt{5}x - 8\sqrt{5}y = 0$

If we let $u = \dfrac{2x-y}{\sqrt{5}}$, $v = \dfrac{x+2y}{\sqrt{5}}$

$\Leftrightarrow \dfrac{(u+1)^2}{1^2} + \dfrac{v^2}{2^2} = 1$.





Rmk · In this example, u and v are chosen so that the u-axis and v-axis are orthogonal.
Such u and v can be found using linear algebra.
· Among the non-degenerate cases, only ellipse is closed and bounded.
$\therefore$ Any continuous $f(x,y)$ restricted to an ellipse has both global max/min.
hyperbola/parabola may not have global max/min.

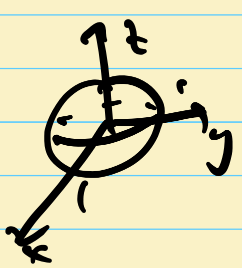
Quadratic constraint for 3-variables

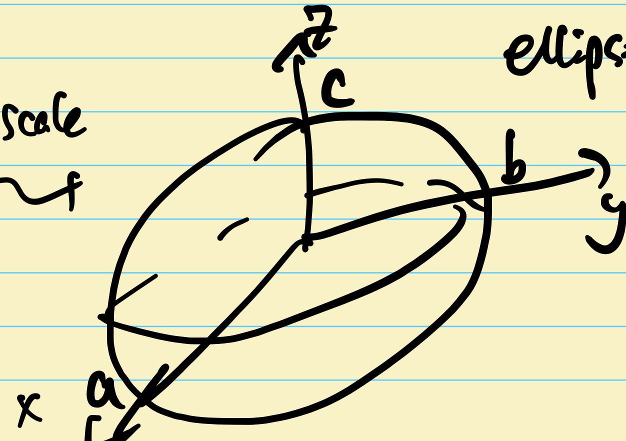
$$g(x,y,z) = Ax^2 + By^2 + Cz^2 + 2Pxy + 2Qyz + 2Rzx + Dx + Ey + Fz + G$$

Typical examples of $g = c$.

· $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1 \quad (a, b, c > 0)$

$x^2 + y^2 + z^2 = 1$ rescale $\rightsquigarrow$ ellipsoid





- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

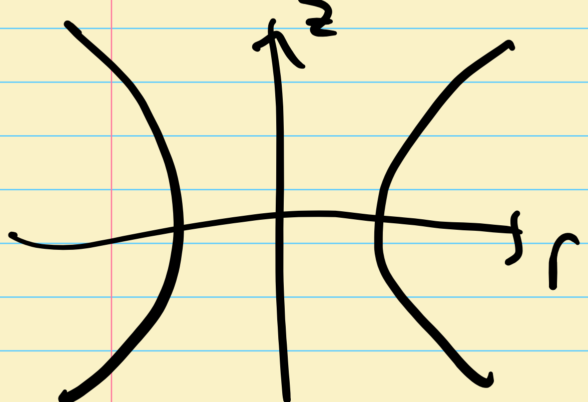
rescale

$x^2 + y^2 - z^2 = 1$

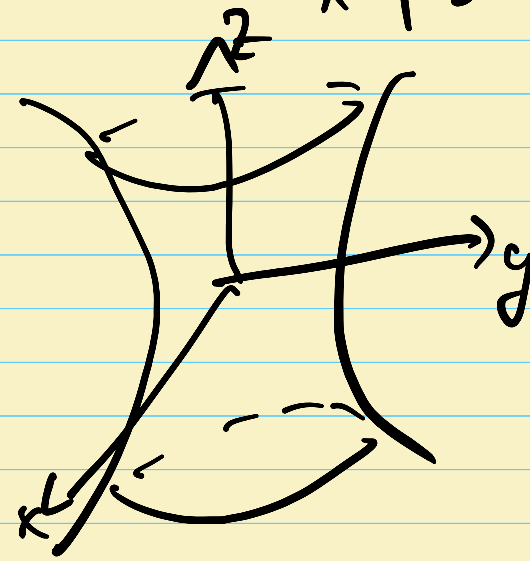
Let $r = \sqrt{x^2+y^2}$ = distance from z-axis

$r^2 - z^2 = 1$

$x^2+y^2-z^2=1.$



rotation around z-axis



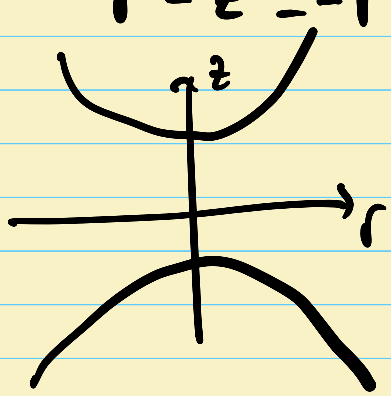
hyperbola

hyperboloid of 1-sheet.

- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1$

$x^2+y^2-z^2=-1$

$r^2 - z^2 = -1$



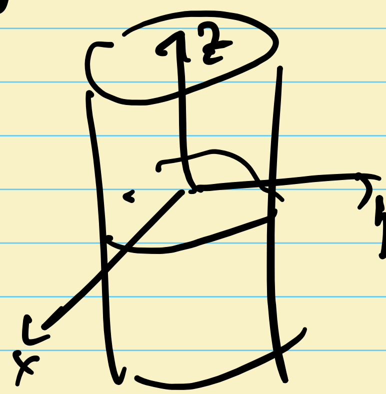
$x^2+y^2-z^2=-1$

hyperbola

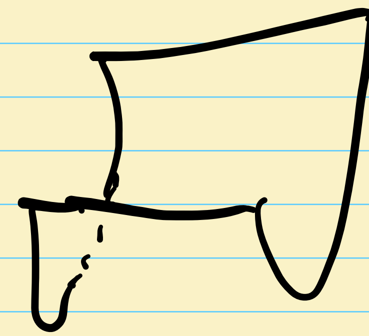
hyperboloid of 2-sheets.

- $x^2+y^2-z^2=0$ (elliptical cone)
- $z=x^2+y^2$ (elliptical paraboloid)
- $z=x^2-y^2$ (hyperbolic paraboloid)

degenerate cases: eg $x^2+y^2=1$ $\qquad$ $z=x^2$



Cylinder of ellipse



Cylinder of parabola

and more degenerate cases.

Similar Fact (2 variable)

Any quadratic constraint $g(x,y,z)=C$ can be transformed to one of standard forms by a change of coordinates.

Rmk Among the above examples, only ellipsoid is closed and bounded.

$\therefore$ Any continuous $f(x,y,z)$ restricted to an ellipsoid has both global max/min.

eg Find the point on the ellipse $x^2+xy+y^2=9$ with maximum x-coordinate.

i.e. $f(x,y)=x$, $g(x,y)=x^2+xy+y^2$

we want to maximize f under constraint $g=9$.

(S·1) By EVT, max of f exist on $g=9$.

$\nabla f=(1,0)$

$\nabla g=(2x+y,\ x+2y)$ $\quad$ $\nabla g=0 \Leftrightarrow x=y=0$

$\neq 0$ on $\{g=9\}$ $\quad$ $(0,0)\notin\{g=9\}$

$$\begin{cases}\nabla f=\lambda\nabla g\\ g=9\end{cases} \Rightarrow \begin{cases}1=\lambda(2x+y) & -① \\ 0=\lambda(x+2y) & -② \\ x^2+xy+y^2=9 & -③\end{cases}$$

$① \Rightarrow \lambda\neq 0 \overset{②}{\Rightarrow} x+2y=0$ i.e. $x=-2y$

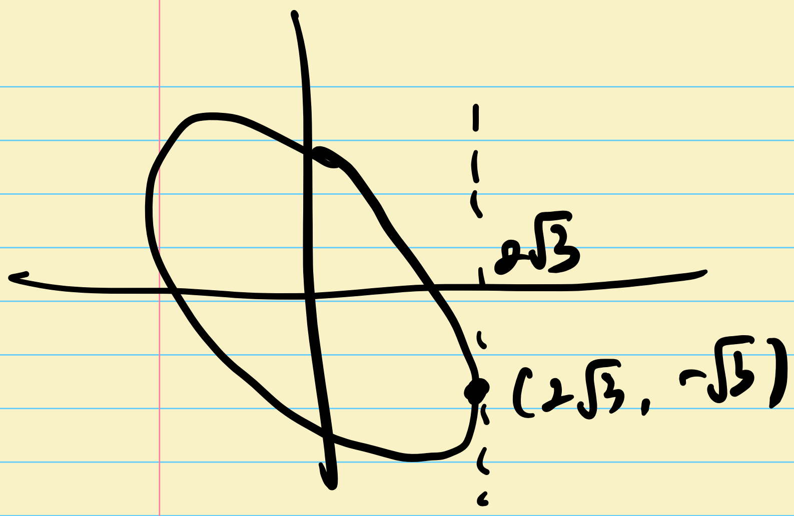
$\overset{③}{\Rightarrow} (-2y)^2+(-2y)y+y^2=9$

$= 3y^2$

$\therefore y=\pm\sqrt{3}$

$\therefore (x,y)=(-2\sqrt{3},\sqrt{3})$ or $(2\sqrt{3},-\sqrt{3})$

max of $f(x,y)=x$ is $2\sqrt{3}$ at $(2\sqrt{3},-\sqrt{3})$



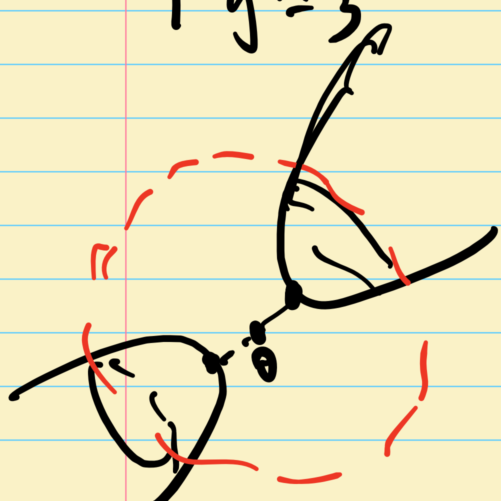
eg Find the points on the hyperboloid $xy - yz - zx = 3$ closest to the origin.

(sol) Let $f(x,y,z) = x^2+y^2+z^2$ = (distance from origin)^2
$g(x,y,z) = xy - yz - zx$.

Minimize f under constraint $g=3$.

$\nabla f = (2x, 2y, 2z)$
$\nabla g = (y-z, x-z, -x-y) \neq 0$ on $g=3$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 3 \end{cases} \iff (x,y,z) = \pm(1,1,-1), \quad \lambda = 1.$$



$f(1,1,-1) = f(-1,-1,1) = 3$

$\therefore$ closest points are $\pm(1,1,-1)$, distance $\sqrt{3}$

Lagrange multiplier with multiple constraints

Let $f, g_1, \cdots, g_k$ be C^1-functions on $\Omega \subseteq \mathbb{R}^n$

$$S = \{x \in \Omega \mid g_i(x) = c_i \quad i = 1, \cdots, k\}$$

Suppose ① a is a local extremum of f on S

② $\nabla g_1(a), \cdots, \nabla g_k(a)$ are linearly independent.

Then $\begin{cases} \nabla f(a) = \sum_{i=1}^{k} \lambda_i \nabla g_i(a) & \text{for some } \lambda_1, \cdots, \lambda_k \in \mathbb{R} \\ g_i(a) = c_i & \text{for } i = 1, \cdots, k. \end{cases}$

eg Maximize $f(x,y,z) = x^2 + 2y - z^2$ on the line L $\begin{cases} 2x - y = 0 \\ y + z = 0 \end{cases}$ in $\mathbb{R}^3$.

[Suppose it is given that f has maximum on L]

(sol) Let $g_1(x,y,z) = 2x - y$, $g_2(x,y,z) = y + z$

$\nabla f = (2x, 2, -2z)$

$\nabla g_1 = (2, -1, 0)$

$\nabla g_2 = (0, 1, 1)$

$(2x, 2, -2z) = \lambda_1 (2, -1, 0) + \lambda_2 (0, 1, 1)$

By Lagrange multipliers, $\begin{cases} \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 = 0 \\ g_2 = 0. \end{cases}$

$$\Rightarrow \begin{cases} 2x = 2\lambda_1 & ① \\ 2 = -\lambda_1 + \lambda_2 & ② \\ -2z = \lambda_2 & ③ \\ 2x - y = 0 & ④ \\ y + z = 0 & ⑤ \end{cases}$$

④,⑤ $\Rightarrow 2x = y = -z$

①,② $\Rightarrow \lambda_1 = x,\ \lambda_2 = -2z$

② $\Rightarrow$
$$\begin{aligned} 2 &= -\lambda_1 + \lambda_2 \\ &= -x - 2z \\ &= -x + 4x \\ &= 3x \end{aligned}$$

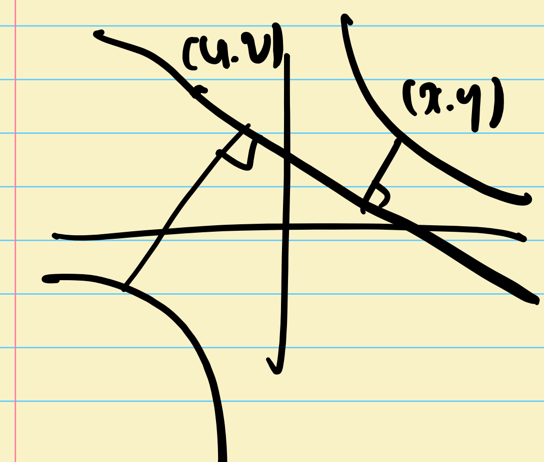
$\Rightarrow x = \frac{2}{3}$

$\Rightarrow y = \frac{4}{3},\quad z = -\frac{4}{3}$

Since solution is unique and we know maximum exists, it must occurs at $\left(\frac{2}{3}, \frac{4}{3}, \frac{4}{3}\right)$ and $f(\ \cdot\cdot\ ) = \frac{4}{3}$.

eg Find the distance between

$C: xy = 1$, $L: x + 4y = \frac{15}{8}$



(sol)

$f(x.y.u.v) = (x-u)^2 + (y-v)^2$

we want to minimize f under constraints

$g_1(x,y,u,v) = xy = 1$

$g_2(\quad " \quad) = u+4v = \frac{15}{8}$

$\nabla f = (2(x-u) \quad 2(y-v) \quad -2(x-u) \quad -2(y-v))$

$\nabla g_1 = (\, y \quad x \quad 0 \quad 0 \,)$

$\nabla g_2 = (\, 0 \quad 0 \quad 1 \quad 4 \,)$

Note that ∇g_1 and ∇g_2 are colinear

$\Leftrightarrow x = y = 0.$ $\quad (x,y) = (0,0) \notin \{xy = 1\}$

$\therefore \nabla g_1, \nabla g_2$ are linearly independent on $\begin{cases} g_1 = 1 \\ g_2 = 15/8 \end{cases}$

Using Lagrange multipliers;

$$\begin{cases} \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 = 1 \\ g_2 = 15/8 \end{cases} \Rightarrow \begin{cases} 2(x-u) = \lambda_1 y & -① \\ 2(y-v) = \lambda_1 x & -② \\ -2(x-u) = \lambda_2 & -③ \\ -2(y-v) = 4\lambda_2 & -④ \\ xy = 1 & -⑤ \\ u+4v = 15/8 & -⑥ \end{cases}$$

case 1) If $\lambda_1 = 0$ or $\lambda_2 = 0$, then

$\overset{①-④}{\Rightarrow}$ $x = u$, $y = v$.

$\overset{⑥}{\Rightarrow}$ $x + 4y = \frac{15}{8}$ i.e. $x = \frac{15}{8} - 4y$

$\overset{⑤}{\Rightarrow}$ $\left(\frac{15}{8} - 4y\right)y = 1$ $\Rightarrow$ no solution.

case 2) If $\lambda_1, \lambda_2 \neq 0$

$\frac{③}{④} = \frac{1}{4} = \frac{x-u}{y-v} = \frac{y}{x}$ $\Rightarrow$ $x = 4y$

$\overset{⑤}{\Rightarrow}$ $4y^2 = 1$ $\Rightarrow$ $y = \pm\frac{1}{2}$ $\therefore$ $(x,y) = \left(2, \frac{1}{2}\right)$ or $\left(-2, -\frac{1}{2}\right)$

If $(x,y) = \left(2, \frac{1}{2}\right)$, $\frac{2-u}{\frac{1}{2}-v} = \frac{1}{4}$ $\Rightarrow$ $8 - 4u = \frac{1}{2} - v$

$$\Rightarrow \begin{cases} -4u + v = -\frac{15}{2} \\ ⑥;\ u + 4v = \frac{15}{8} \end{cases} \Rightarrow (u, v) = \left(\frac{15}{8}, 0\right)$$

If $(x,y)=(-2,-\frac{1}{2}) \underset{\text{similar}}{\Rightarrow} (u,v)=\left(-\frac{225}{136}, \frac{15}{17}\right)$

Compare two solutions:

$f(2,\frac{1}{2},\frac{15}{8},0) = 2^2+(\frac{1}{2})^2+(\frac{15}{8})^2+0^2$ $\wedge$

$f(-2,-\frac{1}{2},-\frac{225}{136},\frac{15}{17}) = 2^2+(\frac{1}{2})^2+\left(\frac{225}{136}\right)^2+\left(\frac{15}{17}\right)^2$

f attains minimum at $(x,y,u,v)=(2,\frac{1}{2},\frac{15}{8},0)$

distance $=\frac{\sqrt{17}}{8}$.

Implicit function theorem

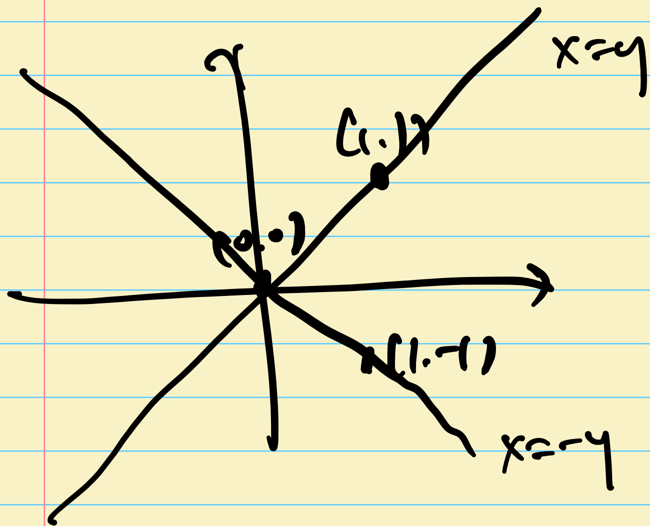
Recall Implicit differentiation

$x^2+y^2=1$. near $(\frac{3}{5},\frac{4}{5})$: y is a function of x $(y=\sqrt{1-x^2})$ and $2x+2y\cdot\frac{dy}{dx}=0$.

↳ when such things possible?

If $g(x,y)=c$, can we find $y=h(x)$ s.t. $g(x,h(x))=c$.

eg Consider level set of $g(x,y) = x^2 - y^2 = 0$



Near $(1,1)$, $y = h(x) = x$

$(1,-1)$, $y = h(x) = -x$

Near $(0,0)$, y is not uniquely determined by x.

eg $S: x^2+y^2+z^2 = 2$ in $\mathbb{R}^3$ 2-dim surface.

can we solve $z = h(x,y)$ or $y = k(x,z)$, or $x = \ell(y,z)$?

Focus near $(0,1,1)$:

If possible to solve z as a differentiable function $z = z(x,y)$ near $(0,1,1)$,

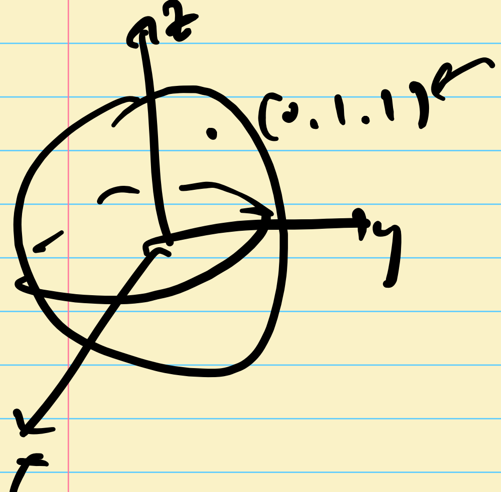
then by implicit differentiation

$$\begin{cases} \dfrac{\partial}{\partial x} : 2x + 2z \cdot \dfrac{\partial z}{\partial x} = 0 \\ \dfrac{\partial}{\partial y} : 2y + 2z \cdot \dfrac{\partial z}{\partial y} = 0 \end{cases} \xRightarrow{(0,1,1)} \begin{matrix} 2\dfrac{\partial z}{\partial x} = 0 \\ 2 + 2 \cdot \dfrac{\partial z}{\partial y} = 0 \end{matrix} \Rightarrow \begin{matrix} \left(\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}\right) \\ \parallel \\ (0,-1) \end{matrix}$$

If possible to solve $x = x(y, z)$ near $(0.1.1)$

$$\begin{cases} \frac{\partial}{\partial y} : 2x \frac{\partial x}{\partial y} + 2y = 0 \\ \frac{\partial}{\partial z} : 2x \frac{\partial x}{\partial z} + 2z = 0 \end{cases} \overset{(0.1.1)}{\Longrightarrow} \begin{cases} 0 + 2 = 0 \\ 0 + 2 = 0 \end{cases}$$ contradiction.

$\therefore$ x is ~~u~~ not a function of y, z near $(0.1.1)$.



near: $z = \sqrt{2 - x^2 - y^2}$

$x = \pm\sqrt{2 - y^2 - z^2}$ not a function on y, z.

Main difference is:

at $(0.1.1)$, $\frac{\partial g}{\partial z} = 2z \neq 0$

$\frac{\partial g}{\partial x} = 2x = 0$

Roughly speaking, implicit function theorem says that $\frac{\partial g}{\partial z}(0.1.1) \neq 0$ guarantees that z is a function of x, y.

eg (multiple constraints)

$$C: \begin{cases} x^2+y^2+z^2=2 & \text{sphere} \\ x+z=1 & \text{plane} \end{cases}$$

Q Is C, $y=y(x)$, $z=z(x)$? possible?

If we can solve y, z as differentiable functions, $y(x), z(x)$.

Implicit differentiation?

$$\frac{\partial}{\partial x} \Rightarrow \begin{cases} 2x+2y\frac{dy}{dx}+2z\frac{dz}{dx}=0 \\ 1+\frac{dz}{dx}=0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} 2y & 2z \\ 0 & 1 \end{pmatrix} \begin{bmatrix} dy/dx \\ dz/dx \end{bmatrix} = \begin{pmatrix} -2x \\ -1 \end{pmatrix}$$

If this linear system

- does not have a solution $\Rightarrow$ $y=y(x)$, $z=z(x)$ DNE.
- have a solution $\Rightarrow$ " may exist.

eg if $(x,y,z) = (0,1,1)$,

$$\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} \begin{vmatrix} dy/dx \\ dz/dx \end{vmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad \begin{matrix} dy/dx = 1 \\ dz/dx = -1 \end{matrix}$$

Implicit functn theorem says that near $(0,1,1)$, $y = y(x)$, $z = z(x)$ is possible because

$\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$ is invertible

<u>Thm</u> (Implicit function theorem)

$\Omega \subseteq \mathbb{R}^{n+k}$ open, $F : \Omega \to \mathbb{R}^k$ be C^1.

Denote (x,y)

where $x = (x_1 \cdots x_n)$

$y = (y_1 \cdots y_k)$

$$F(x,y) = \begin{pmatrix} F_1(x,y) \\ \vdots \\ F_k(x,y) \end{pmatrix}$$

Let $(a,b) \in \Omega$ where $a \in \mathbb{R}^n$, $y \in \mathbb{R}^k$ s.t.

$$F(a,b) = c = \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} \in \mathbb{R}^k.$$

Suppose the $k \times k$ matrix

$$\left[\frac{\partial F_i}{\partial y_j}(a,b)\right]_{\substack{1\le i\le k\\ 1\le j\le k}} = \begin{pmatrix} \frac{\partial F_1}{\partial y_1}(a,b) & \cdots & \frac{\partial F_1}{\partial y_k}(a,b) \\ \vdots & & \vdots \\ \frac{\partial F_k}{\partial y_1}(a,b) & \cdots & \frac{\partial F_k}{\partial y_k}(a,b) \end{pmatrix}$$

is invertible.

Then there exist open sets $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^k$, containing a, b (respectively)

s.t. $\exists$ a unique function $\varphi: U \to V$

with $\varphi(a) = b$ and $F(x, \varphi(x)) = c$

for all $x \in U$. Moreover, φ is C^1.